Task 2

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Exercise 1 We are dealing with a random vector $(X, Y)^T$ with a uniform distribution over the set $M = \{(x, y) : 0 < x < y < 1\}$. Clearly the set M forms the interior of a triangle with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$. Since the distribution is uniform, the density should be constant over its support. The area of the triangle formed by M is clearly $1/2$, so we can conclude that

$$f_{(X,Y)}(x, y) = 2\mathbb{1}_M(x, y).$$

Let us focus on the marginal distributions now. For X we have

$$f_X(x) = \int_{\mathbb{R}} f_{(X,Y)}(x, y)\, dy = \mathbb{1}_{x\in(0,1)} \int_x^1 2\, dy = 2(1 - x)\mathbb{1}_{x\in(0,1)}.$$

For Y we have

$$f_Y(y) = \int_{\mathbb{R}} f_{(X,Y)}(x, y)\, dx = \mathbb{1}_{y\in(0,1)} \int_0^y 2\, dx = 2y\mathbb{1}_{y\in(0,1)}.$$

From marginal distributions, we can clearly see that X and Y are not independent since for example for $(x, y) = (1/2, 1/2)$

$$f_{(X,Y)}(x, y) = 2\mathbb{1}_M(x, y) \neq 2y\mathbb{1}_{y\in(0,1)} 2(1 - x)\mathbb{1}_{x\in(0,1)} = f_X(x)f_Y(y).$$

Exercise 2 To simulate a random sample from the distribution given by (X, Y) we will first consider sampling from the distribution of Y. From the density of Y, we can see that the cumulative distribution function of Y takes the following form on its support.

$$F_Y(y) = y^2, \quad y \in (0, 1)$$

Lets consider a sample $(Z_1, \ldots, Z_n)$ from $U(0, 1)$. Now we can sample from the distribution of Y by taking

$$Y_i = F_Y^{-1}(Z_i).$$

We know that $f_{(X,Y)}$ can be expressed as

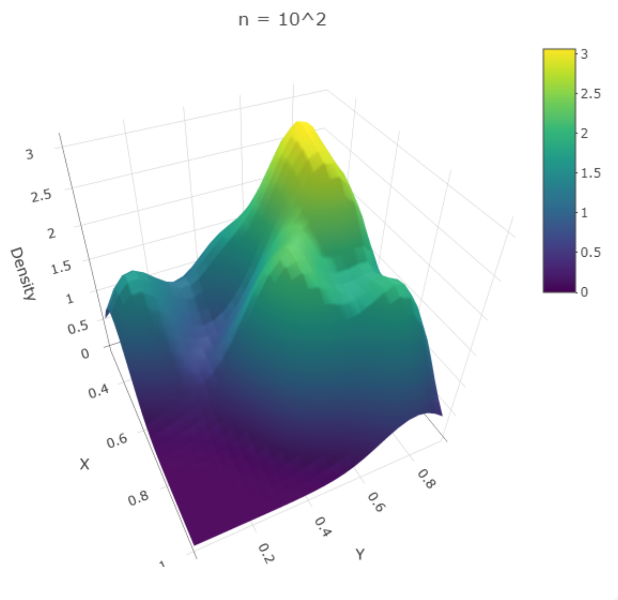
$$f_{(X,Y)}(x, y) = f_{X|Y}(x|y)f_Y(y),$$

where $f_{X|Y}(x|y) = \frac{1}{y}\mathbb{1}_{(0<x<y)}$ on the support of Y. One can recognize that

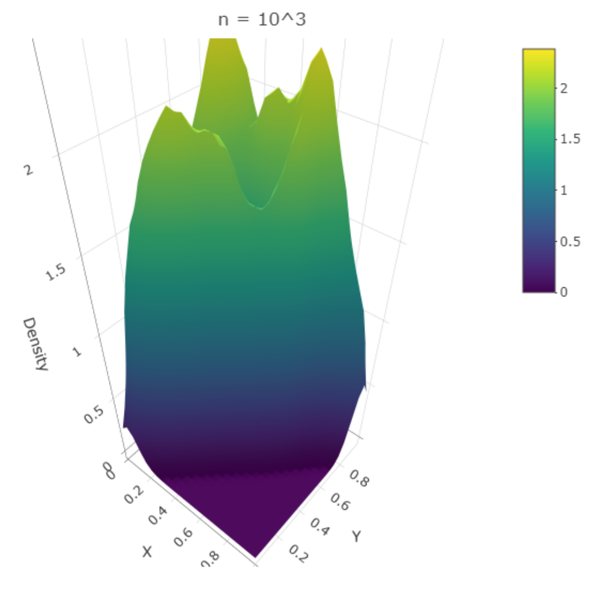
$$X|(Y = y) \sim U(0, y),$$

therefore to simulate from the joint distribution of (X, Y), we can first simulate Y_i as $F_Y^{-1}(Z_i)$ and then simulate corresponding X_i from $U(0, Y_i)$.

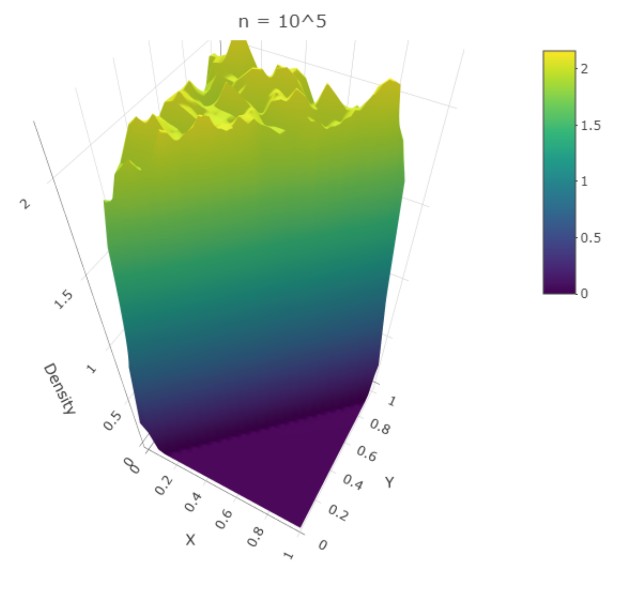
For an empirical evidence that such simulation truly generates from the desirable distribution, we will include figures of estimated joint densities for simulations of sizes $n = 10^2, 10^3, 10^5, 10^6$. One should be able to observe that the estimated densities looks more and more similar to the real density of (X, Y) which is a flat plane above the triangle given by M. Figures are available on the next page.



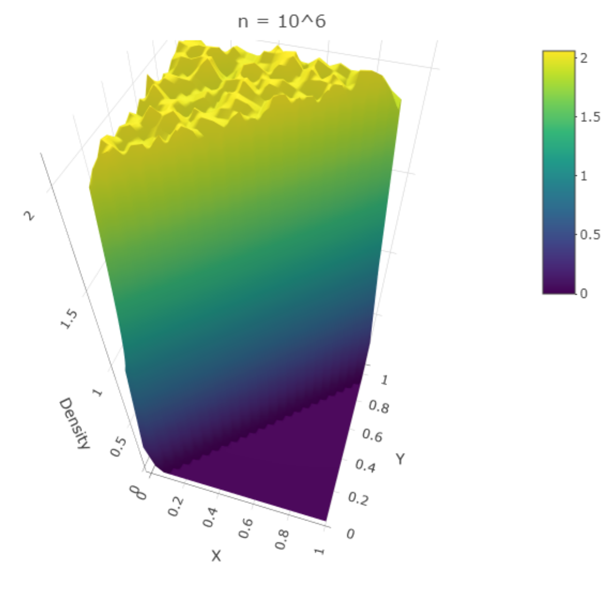
(a) n = 10^2



(b) n = 10^3



(c) n = 10^5



(d) n = 10^6